Devoir de Mathématiques N° 10.

I $A(-1,4)$ $B(-2,-4)$ $D(+2,-2)$, $E(5,2)$

① ABCD parallélogramme $\iff \vec{AB} = \vec{DC}$. $\quad C(x,y)$; $\vec{AB}\begin{pmatrix}-1\\-8\end{pmatrix}$ $\vec{DC}\begin{pmatrix}x-2\\y+2\end{pmatrix}$

$$\iff \begin{cases} x-2 = -1 \\ y+2 = -8 \end{cases}$$

$$\iff \begin{cases} x = 1 \\ y = -10 \end{cases} \quad \text{d'où} \quad C(1,-10)$$

② ABCD a pour centre I milieu de [AC] : $\begin{cases} x_I = (x_A + y_C)\frac{1}{2} = 0 \\ y_I = \frac{y_A + y_C}{2} = -\frac{6}{2} = -3 \end{cases}$ d'où $I(0,-3)$

③ Soit $J(x,y)$, $\vec{JA} = 3\vec{JE}$ $\quad \vec{JA}\begin{pmatrix}-1-x\\4-y\end{pmatrix}$ $\vec{JE}\begin{pmatrix}5-x\\2-y\end{pmatrix}$

$$\iff \begin{cases} -1-x = 3(5-x) \\ 4-y = 3(2-y) \end{cases}$$

$$\iff \begin{cases} 2x = 16 \\ 2y = 2 \end{cases} \iff \begin{cases} x = 8 \\ y = 1 \end{cases} \iff J(8,1)$$

④ $\vec{BD}\begin{pmatrix}4\\+2\end{pmatrix}$ $\vec{BJ}\begin{pmatrix}10\\5\end{pmatrix}$ et $\det(\vec{BD},\vec{BJ}) = 20-20 = 0$

$\implies$ B, D, J alignés.

⑤ Soit $K(x,y)$; $\vec{KA}\begin{pmatrix}-1-x\\4-y\end{pmatrix}$ $\vec{KB}\begin{pmatrix}-2-x\\-4-y\end{pmatrix}$ $\vec{KD}\begin{pmatrix}2-x\\-2-y\end{pmatrix}$

$3\vec{KB} - 4\vec{KD} = 2\vec{KA}$

$$\iff \begin{cases} 3(-2-x) - 4(2-x) = 2(-1-x) \\ 3(-4-y) - 4(-2-y) = 2(4-y) \end{cases} \iff \begin{cases} -6-3x-8+4x = -2-2x \\ -12-3y+8+4y = 8-2y \end{cases}$$

$$\iff \begin{cases} 3x = 12 \\ 3y = 12 \end{cases} \iff \begin{cases} x = 4 \\ y = 4 \end{cases} \iff K(4,4).$$

Autre méthode : $3\vec{KB} - 4\vec{KD} = 2\vec{KA} \overset{\text{Chasles}}{\iff} 3\vec{KO} + 3\vec{OB} - 4\vec{KO} - 4\vec{OD} = 2\vec{KO} + 2\vec{OA}$

$$\iff -3\vec{KO} = 2\vec{OA} - 3\vec{OB} + 4\vec{OD}$$

$$\iff \vec{OK} = \frac{1}{3}(2\vec{OA} - 3\vec{OB} + 4\vec{OD})$$

d'où $\begin{cases} x_K = \frac{1}{3}(2x_A - 3x_B + 4x_D) \\ y_K = \frac{1}{3}(2y_A - 3y_B + 4y_D) \end{cases} \implies K(4,4).$

⑥ $\vec{BK}\begin{pmatrix}6\\8\end{pmatrix}$ $\vec{DE}\begin{pmatrix}3\\4\end{pmatrix}$ $\det(\vec{BK},\vec{DE}) = 24-24=0$

$\Longrightarrow (BK) \parallel (DE)$.

⑦ $\vec{AD}\begin{pmatrix}3\\-6\end{pmatrix}$ $\vec{BD}\begin{pmatrix}4\\2\end{pmatrix}$

$xx'+yy' = 3\times4 - 6\times2 = 0 \Longrightarrow \vec{AD} \perp \vec{BD}$

$\Longrightarrow$ ABD rectangle en D.

(II) ① $\vec{AM}\begin{pmatrix}4m-1\\m\end{pmatrix}$ $\vec{CB}\begin{pmatrix}-3\\1\end{pmatrix}$

$(AM)\parallel(CB) \iff \det(\vec{AM},\vec{CB})=0 \iff 4m-1+3m=0$

$\iff 7m=1$

$\iff m=\frac{1}{7}$

d'où $M\left(\frac{4}{7};\frac{8}{7}\right)$

② CAD rectangle en A $\iff \vec{AC}\perp\vec{AD}$; $D(x,0)$, $\vec{AC}\begin{pmatrix}2\\5\end{pmatrix}$ $\vec{AD}\begin{pmatrix}x-1\\-1\end{pmatrix}$

$\iff xx'+yy'=0$

$\iff 2(x-1)-5=0 \iff 2x=7 \iff x=\frac{7}{2}$

d'où $D(7/2,0)$

③ Soit I milieu de [BC] ; $\begin{cases} x_I = \frac{x_B+x_C}{2} = \frac{3}{2} \\ y_I = \frac{13}{2} \end{cases}$

$M(x,y)\in\Delta \iff \vec{AI},\vec{AM}$ colinéaires $\vec{AI}\begin{pmatrix}1/2\\11/2\end{pmatrix}$ $\vec{AM}\begin{pmatrix}x-1\\y-1\end{pmatrix}$

$\iff \det(\vec{AI},\vec{AM})=0$

$\iff \frac{y-1}{2} - \frac{x-1}{2}\times11 = 0 \iff y-1-11x+11=0$

$\iff y-11x+10=0$.

$\Delta : y=11x-10$.

④ $M(x,y)\in D \iff \vec{CM}\perp\vec{AB}$ $\vec{AB}\begin{pmatrix}-1\\6\end{pmatrix}$; $\vec{CM}\begin{pmatrix}x-3\\y-6\end{pmatrix}$

$\iff xx'+yy'=0$

$\iff -x+3+6y-36=0 \iff -x+6y-33=0$

d'où $D : -x+6y-33=0$

III

① $\vec{AP} = 3\vec{BA} + 3\vec{AP} \iff 2\vec{AP} = -3\vec{BA}$

$\iff \vec{AP} = +\frac{3}{2}\vec{AB}$.

② M symétrique de C par rapport à B

donc $\vec{CM} = 2\vec{CB}$ d'où $\vec{AM} = \vec{AC} + \vec{CM}$

$= \vec{AC} + 2\vec{CB} = \vec{AC} + 2\vec{CA} + 2\vec{AB}$

$= -\vec{AC} + 2\vec{AB}$.

③ On déduit que dans $(A, \vec{AB}, \vec{AC})$

$A(0,0)$; $B(1,0)$; $C(0,1)$, $P(\frac{3}{2},0)$; $M(2;-1)$: $N(\frac{1}{2},2)$

d'où $\vec{MP}\begin{pmatrix}-1/2\\1\end{pmatrix}$ $\vec{MN}\begin{pmatrix}-3/2\\3\end{pmatrix}$

donc $\det(\vec{MP}, \vec{MN}) = -\frac{3}{2} - (-\frac{3}{2}) = 0 \implies$ M, N, P alignés.